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## **Triply special relativity from six dimensions**

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### **Abstract**

We show that the generalization of Doubly Special Relativity to a curved de Sitter background can be obtained starting from a six-dimensional spacetime on which quadratic constraints on position and momentum coordinates are imposed.

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## 1. Introduction

Recently, the possibility of deriving Doubly Special Relativity (DSR) models from higher dimensions has been discussed from several points of view [1-3].

As is well known, DSR models are based on the deformation of the standard energy-momentum dispersion law of relativistic particles [4], that is assumed to be induced by quantum gravity effects at scales near the Planck energy. The deformation is realized by modifying the action of the Lorentz group on momentum space through the introduction of a new observer-independent constant  $\kappa$ , with the dimensions of energy (usually identified with the Planck energy). In this framework, the transformation law of momenta becomes nonlinear, and that of positions momentum dependent. Special relativity is recovered in the limit  $\kappa \rightarrow \infty$ . Different choices of the deformed dispersion law correspond to different DSR models. The various realizations can be obtained choosing different parametrizations of a 5-dimensional momentum space, on which a quadratic constraint analogous to the de Sitter hyperboloid of position space has been imposed [5]. This possibility has induced some authors to derive the DSR dynamics from a five-dimensional model [1]. Another motivation for considering derivations from higher dimensions has been that the transformation laws of some DSR models can be realized linearly in five dimensions [2].

In this paper, we wish to extend the derivation from higher dimensions of doubly special relativity to the case of a de Sitter background spacetime (such a model has been called triply special relativity (TSR) in [6]). The derivation is obtained starting from the phase space of a six-dimensional particle, in a way analogous to that used in [7] to obtain Snyder spacetime [8].

Generalizations of DSR to a (anti)-de Sitter background have been discussed in different contexts in [9-11,6], where various realizations of the deformed de Sitter algebra have been proposed. The main characteristic of these models is that both position and momentum coordinates have nonvanishing Poisson brackets between themselves. Nontrivial Poisson brackets between position coordinates arise naturally in spacetime realizations of DSR models, and lead to noncommutative geometry at the quantum level. Moreover, nontrivial Poisson brackets between momenta (or better, translation generators), are a characteristic of (anti)-de Sitter spacetime.

As we shall see, the phase space of triply special relativity in four dimensions can be obtained in the Hamiltonian formalism, by imposing on the position coordinates a quadratic constraint that generalizes the standard de Sitter hyperboloid, and on the momentum coordinates an analogous quadratic constraint which yields the DSR structure on momentum space [5]. Enforcing both constraints on a 12-dimensional phase space, one is left with one first class and two second class constraints. The hamiltonian reduction of the system gives rise to an 8-dimensional phase space with a symplectic structure adapted to triply special relativity. The specific model obtained depends on the choice of a gauge condition fixing the first class constraint.

## 2. The model

We consider the action

$$S = \int d\tau [\dot{X}_A P_A - (\lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3)], \quad (1)$$

in a six-dimensional space of signature  $(+, -, \dots, -)$ , with  $\lambda_i$  Lagrange multipliers enforcing the constraints

$$\phi_1 = \frac{1}{2}(X_A^2 + \alpha^2), \quad \phi_2 = \frac{1}{2}(P_A^2 + \kappa^2), \quad \phi_3 = X_A P_A. \quad (2)$$

Here  $\alpha$  is the de Sitter radius, while  $\kappa$  is the energy scale of DSR. The field equation read

$$\dot{X}_A = \lambda_2 P_A + \lambda_3 X_A, \quad \dot{P}_A = -\lambda_1 X_A - \lambda_3 P_A. \quad (3)$$

The constraints  $\phi_1 \approx \phi_2 \approx 0$  enforce the de Sitter structure of position and momentum [5] space, while  $\phi_3 \approx 0$  can be considered as a secondary constraint following from the other two.

The constraints satisfy the algebra

$$\{\phi_1, \phi_2\} \approx 0, \quad \{\phi_1, \phi_3\} \approx -\alpha^2, \quad \{\phi_2, \phi_3\} \approx \kappa^2. \quad (4)$$

It is easy to see that they split into one first class constraint  $\psi$  and two second class constraints  $\chi_i$ . The original 12-dimensional phase space can therefore be reduced to 8

independent coordinates. By taking linear combination of the original constraints, one can define

$$\psi = \frac{1}{2}(\kappa^2 X_A^2 + \alpha^2 P_A^2 + 2\alpha^2 \kappa^2), \quad (5)$$

$$\chi_1 = X_A P_A, \quad \chi_2 = \frac{1}{2}(\kappa^2 X_A^2 - \alpha^2 P_A^2), \quad (6)$$

with

$$C_{12} \equiv \{\chi_1, \chi_2\} \approx -2\alpha^2 \kappa^2. \quad (7)$$

In terms of the new variables, the action reads

$$S = \int d\tau \left[ \dot{X}_A P_A - \frac{1}{2} \left( \frac{\lambda_1}{\kappa^2} + \frac{\lambda_2}{\alpha^2} \right) \psi - \frac{1}{2} \left( \frac{\lambda_1}{\kappa^2} - \frac{\lambda_2}{\alpha^2} \right) \chi_1 - \lambda_3 \chi_3 \right]. \quad (8)$$

The compatibility conditions imply that the coefficient of the secondary constraint vanish, and hence

$$\alpha^2 \lambda_1 = \kappa^2 \lambda_2, \quad \lambda_3 = 0. \quad (9)$$

Defining the Dirac brackets of two functions in phase space as  $\{A, B\}^* = \{A, B\} - \{A, \chi_\alpha\} C^{\alpha\beta} \{\chi_\beta, B\}$ , with  $C^{\alpha\beta}$  the inverse of  $C_{\alpha\beta}$ , one has

$$\begin{aligned} \{X_A, X_B\}^* &\approx \frac{1}{2\kappa^2} (X_A P_B - X_B P_A), \\ \{P_A, P_B\}^* &\approx \frac{1}{2\alpha^2} (X_A P_B - X_B P_A), \\ \{X_A, P_B\}^* &\approx \eta_{AB} + \frac{1}{2\alpha^2} X_A X_B + \frac{1}{2\kappa^2} P_A P_B. \end{aligned} \quad (10)$$

The Dirac brackets have a structure analogous to that of the known DSR model in de Sitter space [9-11,6]. In particular the Poisson brackets of positions and of momenta are proportional to the classical generators of Lorentz transformations, while the position-momentum brackets contain terms quadratic in positions and momenta.

However, in order to complete the Hamiltonian reduction and obtain a 8-dimensional phase space adapted to a four-dimensional particle, one must still fix a gauge condition for the first-class constraint. According to [1], this choice singles out a specific model between the possible realizations of DSR in de Sitter space.

We give two examples of gauge conditions: The simplest is  $\theta \equiv P_5 = 0$ . It follows

$$P_4 = \sqrt{P_\mu^2 + \kappa^2}, \quad X_4 = \frac{X_\mu P_\mu}{\sqrt{P_\mu^2 + \kappa^2}}, \quad X_5 = \sqrt{\frac{(X_\mu^2 + \alpha^2)(P_\mu^2 + \kappa^2) - (X_\mu P_\mu)^2}{P_\mu^2 + \kappa^2}},$$

where the greek indices run from 0 to 3. All the constraint are now second class and the matrix  $C_{\alpha\beta}$  reads

$$C_{\alpha\beta} \approx \begin{pmatrix} 0 & 0 & 0 & -\kappa^2 X_5 \\ 0 & 0 & -2\alpha^2 \kappa^2 & -\kappa^2 X_5 \\ 0 & 2\alpha^2 \kappa^2 & 0 & 0 \\ \kappa^2 X_5 & \kappa^2 X_5 & 0 & 0 \end{pmatrix}, \quad (11)$$

with inverse

$$C^{\alpha\beta} \approx \begin{pmatrix} 0 & 0 & -\frac{1}{2\alpha^2 \kappa^2} & \frac{1}{\kappa^2 X_5} \\ 0 & 0 & \frac{1}{2\alpha^2 \kappa^2} & 0 \\ \frac{1}{2\alpha^2 \kappa^2} & -\frac{1}{2\alpha^2 \kappa^2} & 0 & 0 \\ -\frac{1}{\kappa^2 X_5} & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

The Dirac brackets for the 4-dimensional coordinates in this gauge are given by

$$\begin{aligned} \{X_\mu, X_\nu\}^* &\approx \frac{1}{\kappa^2}(X_\mu P_\nu - X_\nu P_\mu), \\ \{P_\mu, P_\nu\}^* &\approx 0 \\ \{X_\mu, P_\nu\}^* &\approx \eta_{\mu\nu} + \frac{1}{\kappa^2}P_\mu P_\nu. \end{aligned} \quad (13)$$

These are the Poisson brackets associated with the Snyder model [8], which were obtained in [7] in a slightly different way. Likewise, the gauge  $X_5 = 0$ , would have produced the de Sitter algebra.

Another possible choice of gauge is  $\theta \equiv \frac{X_5}{\alpha} + \frac{P_5}{\kappa} = 0$ . In this case, the matrix  $C_{\alpha\beta}$  reads

$$C_{\alpha\beta} \approx \begin{pmatrix} 0 & 0 & 0 & 2\kappa X_5 \\ 0 & 0 & -2\alpha^2 \kappa^2 & 0 \\ 0 & 2\alpha^2 \kappa^2 & 0 & -\frac{2X_5}{\alpha} \\ -2\kappa X_5 & 0 & \frac{2X_5}{\alpha} & 0 \end{pmatrix}, \quad (14)$$

with inverse

$$C^{\alpha\beta} \approx \begin{pmatrix} 0 & -\frac{1}{2\alpha^3 \kappa^3} & 0 & -\frac{1}{2\kappa X_5} \\ \frac{1}{2\alpha^3 \kappa^3} & 0 & \frac{1}{2\alpha^2 \kappa^2} & 0 \\ 0 & -\frac{1}{2\alpha^2 \kappa^2} & 0 & 0 \\ \frac{1}{2\kappa X_5} & 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

and the Dirac brackets for the 4-dimensional coordinates are

$$\begin{aligned}\{X_\mu, X_\nu\}^* &\approx \frac{1}{2\kappa^2}(X_\mu P_\nu - X_\nu P_\mu), \\ \{P_\mu, P_\nu\}^* &\approx \frac{1}{2\alpha^2}(X_\mu P_\nu - X_\nu P_\mu), \\ \{X_\mu, P_\nu\}^* &\approx \eta_{\mu\nu} + \frac{1}{2\alpha^2}X_\mu X_\nu + \frac{1}{2\kappa^2}P_\mu P_\nu + \frac{1}{\alpha\kappa}P_\mu X_\nu.\end{aligned}\quad (16)$$

These are the commutation relations proposed in [6] for TSR, in a representation in which the Lorentz generators maintain the classical form  $J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$ .

The previous gauge conditions can be generalized to  $\theta \equiv \mu_1 \frac{X_5}{\alpha} + \mu_2 \frac{P_5}{\kappa} = 0$ , with generic coefficients  $\mu_1, \mu_2$ , normalized so that  $\mu_1^2 + \mu_2^2 = 1$ . Going through the same steps as before, one obtains the Dirac brackets

$$\begin{aligned}\{X_\mu, X_\nu\}^* &\approx \frac{\cos^2 \mu}{\kappa^2} (X_\mu P_\nu - X_\nu P_\mu), \\ \{P_\mu, P_\nu\}^* &\approx \frac{\sin^2 \mu}{\alpha^2} (X_\mu P_\nu - X_\nu P_\mu), \\ \{X_\mu, P_\nu\}^* &\approx \eta_{\mu\nu} + \frac{\sin^2 \mu}{\alpha^2} X_\mu X_\nu + \frac{\cos^2 \mu}{\kappa^2} P_\mu P_\nu + \frac{2 \sin \mu \cos \mu}{\alpha\kappa} P_\mu X_\nu,\end{aligned}\quad (17)$$

where we have defined  $\mu_1 = \cos \mu$ ,  $\mu_2 = \sin \mu$ .

Once one has performed the reduction of the phase space, the 4-dimensional dynamics can be studied in a standard way by imposing a Hamiltonian constraint  $P_\mu^2 = M^2$ , or equivalently  $P_4^2 + P_5^2 = M^2 + \kappa^2$  [1].

According to [1], it should also be possible to define a set of 4-dimensional coordinates commuting with all the constraints, in terms of the 6-dimensional ones. These coordinates would satisfy Poisson brackets identical to the Dirac brackets obtained above. Unfortunately, we were not able to find an explicit expression for them.

### 3. Conclusion

A six-dimensional constrained system has been reduced to four dimensions, giving rise to Dirac brackets compatible with DSR in a de Sitter background (TSR). In limit cases one obtains the Snyder model or de Sitter special relativity. The specific four-dimensional model obtained depends on a choice of gauge (i.e. of projection from six dimensions). In

particular, we have been able to recover the phase space of the model proposed in ref. [6]. Of course, more complicated TSR models [9-11] can be obtained with less simple gauge choices. Clearly, the various models are equivalent from a six dimensional point of view, but not in a four-dimensional perspective.

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